

In addition, we have respectively for the P.D., and exceedance probability, or APD (a posterior probability here, that  $E$  exceeds a level  $E_0(>0)$ ) defined as usual by

$$D_1(E_0) \equiv \int_0^{E_0} W_1(E) dE \quad ; \quad P_1(E \geq E_0) \equiv \int_{E_0}^{\infty} W_1(E) dE = 1 - D(E_0), \quad (2.16)$$

the following results, where we have used

$$\int_0^z z J_0(z) dz = z J_1(z), \quad (2.16a)$$

viz:

$$D_1(E_0) = E_0 \int_0^{\infty} J_1(r E_0) dr \int_0^{2\pi} \hat{F}_1(ir, \phi) d\phi / 2\pi, \quad E_0 \geq 0 \quad (2.17a)$$

$$P_1(E \geq E_0) = 1 - E_0 \int_0^{\infty} J_1(r E_0) dr \int_0^{2\pi} \hat{F}_1(ir, \phi) d\phi / 2\pi. \quad (2.17b)$$

Our results (2.13)-(2.17) are generalizations of earlier results [Furutsu and Ishida, 1960; Middleton, 1972b; Giordano, 1970], where our basic assumptions, so far, postulate only poisson distributions of source location and emissions, e.g. essentially independent sources. No restrictions on the specific character of the statistics of the source parameters are as yet introduced. It is for this reason that the characteristic function  $\hat{F}_1$  depends on  $\phi$ , as well as on  $r$ .

## 2.2 First Reduction of the c.f. $\hat{F}_1$ : The Narrow-Band Receiver Condition

At this point we invoke certain properties of the basic waveform  $B_0 \cos[\phi_s' + \mu_d \omega_0 (\lambda + \hat{\epsilon}) - \omega_0 \epsilon_d t - \phi]$  which appears in the exponent in the integrand of (2.13). We use the facts that (i),  $B_0$ ,  $\phi_s$ , are both slowly-varying functions of  $\lambda$ ; and (ii), the process density  $\rho(\lambda, \hat{\epsilon})$  is likewise slowly varying, vis-à-vis  $\cos \omega_0 \mu_d \lambda$ ,  $\sin \omega_0 \mu_d \lambda$ . Employing the familiar expansion in Bessel functions,

$$\exp[ia \cos \phi] = \sum_{m=0}^{\infty} i^m \epsilon_m J_m(a) \cos m\phi, \quad (2.18)$$

in (2.13), we see that only for  $m=0$  does the integrand (containing the exponent) contribute, as all the other terms are highly oscillatory in regions where  $B_0$ ,  $\phi_s$ , and  $\rho$  are slowly changing with  $\lambda$ . The result is the important simplification of (2.13) to

$$\hat{F}_1(ir, \phi) = \exp \left\{ \left\langle \int_{\Lambda} \rho(\lambda, \hat{e}) [J_0(rB_0[t, \lambda | \hat{e}, \theta] - 1)] d\lambda d\hat{e} \right\rangle_{\theta} \right\} = \hat{F}_1(ir), \quad (2.19)$$

which is valid, provided that  $U_{nb}$  is truly narrow-band, e.g.  $\Delta f_{ARI} \ll f_0$ : the (composite) bandwidth of the (linear) aperture-RF-IF receiver stages is much less than the (IF) central frequency  $f_0$ . [The result (2.19) is now in the same form as obtained in earlier work, but still somewhat more general in detail.]

The important analytic feature of (2.19) is that now, because of the narrow-band receiver condition  $\Delta f_{ARI} \ll f_0$ , the c.f.  $\hat{F}_1$  is independent of  $\phi$ . Accordingly, we see that (2.14), (2.15), (2.17) reduce at once to the simpler forms (with the help of (2.18):

$$w_1(E, \psi) = \frac{E}{2\pi} \int_0^{\infty} r J_0(rE) \hat{F}_1(ir) dr = W_1(E) W_1(\psi), \quad (2.20)$$

with

$$W_1(E) = E \int_0^{\infty} r J_0(rE) \hat{F}_1(ir) dr; \quad W_1(\psi) = 1/2\pi, \quad 0 \leq \phi < 2\pi, \quad (2.21)$$

and

$$D(E_0) = E_0 \int_0^{\infty} J_1(rE_0) \hat{F}_1(ir) dr, \quad (2.22a)$$

$$P(E > E_0) = 1 - E_0 \int_0^{\infty} J_1(rE_0) \hat{F}_1(ir) dr. \quad (2.22b)$$

Respectively for the PD and exceedance probability,  $P$ , for these narrow-band interference waves (in the n.b. receiver), the first-order p.d. of phase  $\psi$  is seen to be uniform over an (IF) cycle ( $T_0 = 1/f_0$ ). The results (2.20)-(2.22b) are formally identical to those derived by Furutsu and Ishida [1960, Eqs. (2.9)-(2.11)], and by Giordano [1970], and, Giordano and Haber [1972], for example. Furthermore,  $\hat{F}_1(ir)$  is clearly a Hankel transform of  $W_1(E)$ , from (2.21) and the fact that the inverse of (2.21) is

$$\hat{F}_1(ir) = \int_0^\infty J_0(rE) W_1(E) dE \equiv \langle J_0(rE) \rangle_E \quad (2.23)$$

[This relation is easily established with the help of

$$\sqrt{ab} \int_0^\infty J_m(ax) J_m(bx) x dx = \delta(b-a), \quad (2.24)$$

cf. p. 943, Morse and Feshbach [1953], applied to  $\langle J_0(rE) \rangle_E$ , with (2.21) for  $W_1(E)$ .] An equivalent expression, now in terms of the average over  $E$  and  $\psi$ , is obtained at once from the fact that

$$J_0(rE) = \int_0^{2\pi} e^{irE \cos \psi} \frac{d\psi}{2\pi} = \int_0^{2\pi} e^{irE \cos \psi} W_1(\psi) d\psi = \langle e^{irE \cos \psi} \rangle_\psi, \quad (2.24a)$$

[cf. (2.2.)], and from the relation (2.23), viz:

$$\hat{F}_1(ir) = \langle e^{irE \cos \psi} \rangle_{E, \psi},$$

(2.24b)

from which is seen the fact that  $\hat{F}_1$  here is also the joint c.f. of envelope and phase, as expected. We shall use this result later in the calculation of moments, (cf. Sec. 5.2).

Next, let us look at the process density  $\rho(\underline{\lambda}, \hat{\epsilon})$ : we write

$$(2.25) \quad \left\{ \begin{array}{l} \rho(\underline{\lambda}, \hat{\epsilon}) = \rho_{\Lambda}(\underline{\lambda} | \hat{\epsilon}) v_T(\hat{\epsilon}): \text{ (av. no. of emitting sources (in } \Lambda) \text{ per} \\ \text{unit domain) and emitting at } \hat{\epsilon}, \text{ in the} \\ \text{interval } d\hat{\epsilon}) \times \text{(av. no. of emissions,} \\ \text{per source, per interval } d\hat{\epsilon}) \text{ in the} \\ \text{observation interval } T; \\ \\ = \rho_{\Lambda}(\underline{\lambda}) v_T(\hat{\epsilon}): \text{ av. no. of emissions per unit domain (d}\Lambda) \\ \text{and per interval } d\hat{\epsilon} \text{ in } T. \text{ This last is on} \\ \text{the reasonable assumption that location} \\ \text{and emission are independent "events".} \end{array} \right.$$

Then, we observe further that

$$\rho_{\Lambda}(\underline{\lambda}) = \sigma_{\Lambda}(\underline{\lambda}) |J_{\Lambda}(\underline{\lambda})| ; \quad \left. \begin{array}{l} |J_{\Lambda}| = c^2 \lambda \text{ for surfaces} \\ = c^3 \lambda^2 \sin \theta, \text{ for volumes,} \end{array} \right\} \begin{array}{l} \text{[Sec. 2.3,} \\ \text{Middleton} \\ \text{(1974)] ,} \end{array} \quad (2.26)$$

in which  $\sigma_{\Lambda}$  is the physical density of emitting sources in the source domain  $\Lambda$ . We now define

$$\left\{ \begin{array}{l} A_{\Lambda} \equiv \int_{\Lambda} \rho_{\Lambda}(\underline{\lambda}) d\underline{\lambda}: \text{ av. no. of emitting sources in } \Lambda; \quad (2.27a) \\ \\ A_{\epsilon, T} \equiv \int_T v_T(\hat{\epsilon}) d\hat{\epsilon}: \text{ av. no. of emissions (per source) in the observa-} \\ \text{tion period, } T. \quad (2.27b) \\ \\ A_T \equiv \int_{\Lambda, T} \rho(\underline{\lambda}, \hat{\epsilon}) d\underline{\lambda} d\hat{\epsilon} = A_{\Lambda} A_{\epsilon, T}: \text{ av. no. of emissions in the period } T. \quad (2.27c) \end{array} \right.$$

Consequently, we can also define probability densities for source location and emission by

$$w_I(\underline{\lambda}) \equiv \rho_{\Lambda}(\underline{\lambda}) / A_{\Lambda} ; \quad w_I(\hat{\epsilon})_T \equiv v_T(\hat{\epsilon}) / A_{\epsilon, T}. \quad (2.28)$$

Now let us look at the integrand,  $\hat{I}$ , of (2.19), and use (2.25)-(2.28) to write

$$\hat{I}_T = A_\Lambda A_{\epsilon, T} \left\langle \int_{\Lambda, \hat{\epsilon}} w_1(\underline{\lambda}) w_1(\hat{\epsilon}) J_0(r B_0[t - \lambda - \hat{\epsilon}; \underline{\lambda}, \underline{\theta}]) - 1 \, d\underline{\lambda} d\hat{\epsilon} \right\rangle, \quad (2.29)$$

where explicitly we have from earlier work [Middleton, 1972b, 1974] for the received envelope  $B_0$  of a typical emission

$$B_0 = |a_R(\underline{\lambda}, f_0) a_T(\underline{\lambda}, f_0)| A_{OT}(t - \lambda - \hat{\epsilon} | \underline{\theta}) g(\underline{\lambda}), \quad (2.30)$$

where

$$\left. \begin{aligned} a_R, a_T &= (\text{complex}) \text{ beam patterns of receiver and typical} \\ &\quad \text{interfering source;} \\ A_{OT} &= (\text{real}) \text{ envelope of the source emission;} \\ g(\underline{\lambda}) &= \text{a geometric factor, which describes the propagation law,} \\ &\quad \text{from source to receiver (which are assumed to be in each} \\ &\quad \text{other's far field).} \end{aligned} \right\} \quad (2.30a)$$

[For this receiver, although the aperture may be comparatively broad-band, as may be that of the source, it is the narrowest filter, of the combination (aperture  $\times$  RF  $\times$  IF) which is controlling. By assumption, one or more of these filters is very narrow vis-à-vis  $f_0$ , cf. the comments following (2.19), so that the effective aperture response here is determined essentially by the response at (and about)  $f_0$ , cf. (2.30).]

Next we let

$$\begin{aligned} T_S &= \text{duration of a typical emission, at the IF output} \\ &\quad (\text{which may be } \leq T); \end{aligned} \quad (2.31a)$$

this can generally be a random variable (one of the  $\theta$  in (2.29)). We now let

$$t - \lambda - \hat{\epsilon} \equiv y = \bar{T}_S z, \quad (2.31b)$$

where  $\bar{T}_S$  is the mean duration of an emission (at the IF output), and  $z$  is dimensionless, to rewrite (2.29) as

$$\hat{I}_T = A_\Lambda A_{\epsilon, T} \bar{T}_s \left\langle \int_0^{T_s/\bar{T}_s} w_1(t - \lambda - \bar{T}_s z) \hat{\epsilon} \{ J_0[r \hat{B}_0(z, \lambda; \theta')] - 1 \} \right\rangle_{T_s, \lambda, \theta'} dz \quad (2.32)$$

all  $(t \in T)$ .

Here  $\theta'$  are any other random parameters in  $\hat{B}_0$ , e.g. amplitude of the basic emission envelope  $B_0$  (in the receiver, at output of IF);  $\hat{B}_0$  itself is  $B_0(z/\bar{T}_s, \lambda; \theta')$ . Further reduction is obtained by writing

$$A_{OT}(\bar{T}_s z | \theta) = A_0 e_{OY} u_0(z), \quad (u_0(z) = 0, z > T_s/\bar{T}_s, z < 0) \quad (2.33)$$

where

$$\begin{cases} A_0 & = \text{(peak) amplitude of the received envelope (at output of the IF);} \\ e_{OY} & = \text{a limiting "voltage" setting (in suitable dimensions), at which} \\ & \quad \text{the receiver will respond to a test signal, above the receiver} \\ & \quad \text{noise,* at output of the IF;} \\ u_0(t) & = \text{normalized envelope wave form at output of receiver IF.} \end{cases} \quad (2.34)$$

Note from (2.33) that the generic waveform  $u_0(z)$  is, of course, required to vanish outside the time interval during which the typical emission ( $\sim A_0 e_{OY} u_0$ ) is "on", e.g. for  $\bar{T}_s z > T_s$ ,  $\bar{T}_s z < 0$ .

Finally, let us multiply and divide  $\hat{I}$  by  $T$ , the observation period, and write (2.32) as

$$\hat{I}_T = \left( \frac{A_\Lambda A_{\epsilon, T}}{T} \right) \bar{T}_s \left\langle \int_0^{T_s/\bar{T}_s} \{ T w_1(t - \lambda - z \bar{T}_s) \hat{\epsilon} \} [J_0(r \hat{B}_0(z, \lambda | \theta')) - 1] \right\rangle_{\theta', \lambda, T_s} dz, \quad (2.34)$$

all  $(t \in T)$ . Also, we see that

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\* The precise definition of  $e_{OY}$  can be determined by standard decision-theoretical techniques of detection [Middleton, 1960; Chapters 18, 19], with some appropriate choice of false alarm rate.

$$\therefore \begin{cases} A_{\Lambda} A_{\epsilon, T} / T \equiv \bar{v}_T: & \text{av. no. of emissions per second, in the} \\ & \text{observation period } T, \\ v_T \bar{T}_S \equiv \gamma_T: & \text{"density" of the process [cf. Sec. 11.2; Eq.} \\ & \text{(11.74), Middleton, 1960]: (av. no. of emis-} \\ & \text{sions per second) \times (\text{mean duration of an emis-} \\ & \text{sion).} \end{cases} \quad (2.35a)$$

$$(2.35b)$$

Equation (2.34) is a generalization of earlier results, which permits the treatment of nonstationary régimes.

At this point we restrict our attention to the most common situation of "local stationarity", whereby it is assumed that there are no changes in average source numbers and emission properties during the observation period  $T$ , and that the emission probability  $w_1(\hat{\epsilon})$  is uniform, e.g.  $Tw_1 - \hat{\epsilon} = 1$ , for all allowed values of  $z$ . Thus, (2.34) reduces to the basic form

$$[\text{uniform p.d. of } \hat{\epsilon}]: \quad \hat{I}_T(r) = \bar{v}_T \bar{T}_S \left\langle \int_0^{T_S / \bar{T}_S} [J_0(r \hat{B}_0) - 1] dz \right\rangle_{\omega, T_S, \omega'}, \quad (2.36)$$

which is the one from which we develop our subsequent analysis (beginning with (2.38)) in this Report. Furthermore, for the idealized steady-state situation where  $T \rightarrow \infty$ , we write

$$\left. \begin{aligned} \lim_{T \rightarrow \infty} \bar{v}_T &= \bar{v}_{\infty} \quad (= A_{\Lambda} \lim_{T \rightarrow \infty} (A_{\epsilon, T} / T)) \\ \lim_{T \rightarrow \infty} \gamma_T &= \gamma_{\infty} \end{aligned} \right\} \quad (2.37)$$

and, accordingly, (2.36) becomes

[uniform p.d. of  $\hat{\epsilon}$ ]:

$$\hat{I}_{\infty}(r) = A_{\infty} \left\langle \int_0^{T_S / \bar{T}_S} [J_0(r \hat{B}_0) - 1] dz \right\rangle_{\omega, T_S, \omega'} ; \quad A_{\infty} \equiv \gamma_{\infty} = \bar{v}_{\infty} \bar{T}_S :$$

$$(2.38)$$

This limiting form of (2.36) is the expression which we shall exploit in the remainder of the study.

The quantity  $A_{\infty}$  appearing in (2.38) is

$$A_{\infty}(=\gamma_{\infty}): \text{ impulsive index (of the present analysis)*} \quad (2.39)$$

As we have already noted in our earlier studies [Middleton, 1972b, 1973, 1974], the Impulsive Index is a measure of the temporal "overlap" or "density", at any instant, of the superposed interference waveforms at the receiver's IF output. It is one of the key parameters of the interference model, in that it critically influences the character of the p.d.'s and P.D.'s of the interference, as observed at the output of the initial (linear) stages of a typical narrow-band receiver. With small values of  $A_{\infty}$  the statistics of the resultant output waveform are dominated by the overlapping of comparatively few, deterministic waveforms, of different levels and shapes, so that the interference has an "impulsive", somewhat structured appearance. For increasingly large values of  $A_{\infty}$  the resultant approaches a normal, or gaussian process, as one would expect from the Central Limit Theorem [Middleton, 1960, Sec. 7.7], as we shall see in more detail later [cf. Sec. 2.4].

### 2.3 Interference Classes A, B, and C: The Rôle of Input and Receiver Bandwidths:

We are now ready to examine the basic form, (2.38), of  $\hat{I}_{\infty}(r)$  [=log  $\hat{F}_1(ir)$ ]. The rôle of the duration  $T_s$  of a typical emission (as perceived at the output of the ARI ( $\equiv$  aperture x RF x IF) stages of the narrow-band receiver) is critical in determining the form of  $\hat{I}_{\infty}(r)$ .

Let us consider first the important special case when the emission duration  $T_s$  is fixed. From Eqs. (2.63a,b), (2.70), (2.72a) of Middleton [1960] we may write for the envelope  $\hat{B}_0$ , cf. (2.10), (2.30), (2.31a), (2.33),

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\*  $A_T (\equiv A_{\Lambda} A_{\epsilon, T} = \bar{\gamma}_T T$ , cf. (2.27c), (2.35)) was designated "impulsive index",  $A$ , in the author's earlier treatments [Middleton, 1972b, 1973, 1974].